MAC-CPTM Situations Project

Situation 02: Parametric Drawings

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Prompt

This example, appearing in CAS-Intensive Mathematics (Heid and Zbiek, 2004)[1], was inspired by a student mistakenly grabbing points representing both parameters (A and B in f(x) = Ax + B) and dragging them simultaneously (the difference in value between A and B stays constant). This generated a family of functions that coincided in one point. Interestingly, no matter how far apart A and B were initially, if grabbed and moved together, they always coincided on the line x = -1.

Commentary

In this case, GSP was a vehicle that brought mathematical relationships to the fore. When one sees such a phenomenon, one can enhance their experience by noticing the potential for mathematics in the patterns that are seen.

Focus 1 will show how the graphical phenomenon can be explained using a symbolic proof. Focus 2 will then show the extension of the phenomenon to quadratic functions (which also appears in CAS-Intensive Mathematics). This focus will show how this can also extend to a polynomial of higher degree (which generated another interesting relationship along with its proof).

Mathematical Foci

Mathematical Focus 1

Graphical phenomenon can frequently be explained using symbolic proof.

Figure 1 displays a screen dump after A and B have been simultaneously dragged.

[1] Heid, M. K. & Zbiek, R. M. (2004). The CAS-Intensive Mathematics Project. NSF Grant No. TPE 96-18029

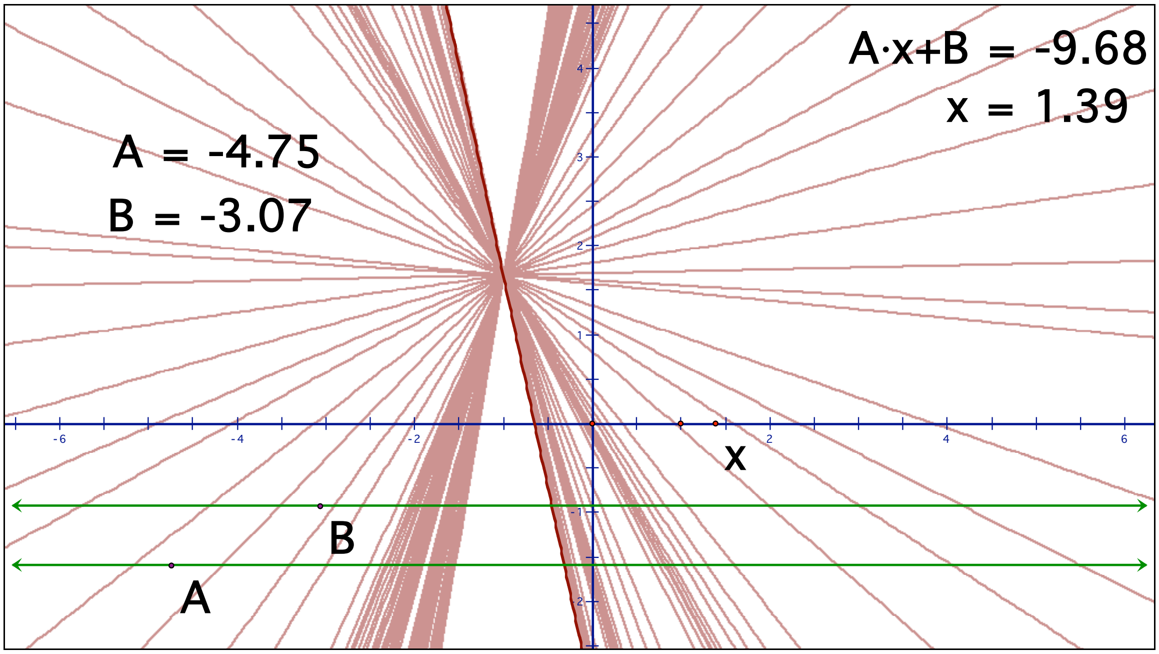


Figure 1. Screen dump showing trace of $f(x) = Ax + B$ after A and B have been dragged simultaneously.

Notice that the family of lines that appear intersect on the line x = -1. To explain why this will be the case for any value of A and B where their difference remains constant can be explained using the following symbolic proof:

Let y = Ax + B and suppose A – B = k, a constant. Let $y_1 = A_1x + B_1$ and $y_2 = A_2x + B_2$ be two lines in this family. $A_1 - B_1 = k = A_2 - B_2 \Rightarrow A_1 - A_2 = B_1 - B_2$. Thus, to determine the point of intersection, y_1 is set equal to y_2. Therefore,

$$A_1x + B_1 = A_2x + B_2$$

$$(A_1 - A_2)x = B_2 - B_1$$

$$x = \frac{B_2 - B_1}{A_1 - A_2} = \frac{-(B_1 - B_2)}{A_1 - A_2} = -1$$

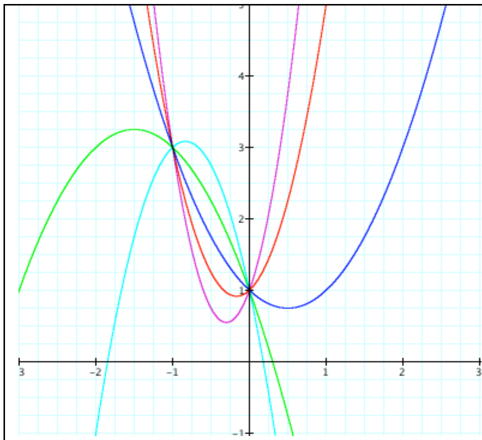
Hence, and two lines in this family will intersect at the point (-1, k).

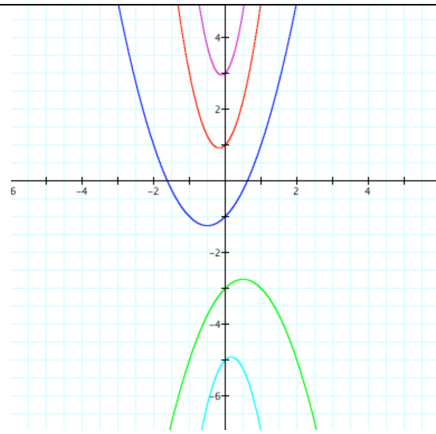
Mathematical Focus 2

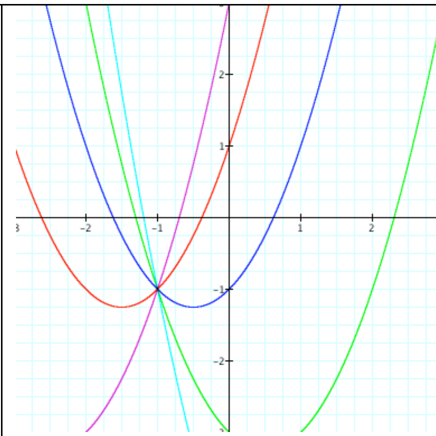
Many times phenomena that are observed for certain functions can be extended to other functions with similar properties, i.e. linear functions extended to polynomial functions with degree > 1.

When looking at quadratic functions, additional assumptions must be made to investigate the phenomenon. In a quadratic function, there are three coefficients rather than two (like in the linear function). For example, if $y = Ax^2 + Bx + C$, there are three possibilities to consider. Either we hold C constant with A – B = k, B constant with A – C = k, or A constant with B – C = k. In each of these cases, similar symbolic proofs as the one in Focus 1 can be worked and the following conclusions can be drawn:

1. If C is held constant, there will be two intersections at (0, C) and (-1, k + C).
2. If B is held constant, there will be no intersection because the equations reduce to $x^2 = -1$.
3. If A is held constant, there will be one intersection at (-1, A – k)


C = 1, A – B = 2


B = 1, A – C = 2


A = 1, B – C = 2

This idea can now be extended for polynomials of the n^{th} degree. In order to do this extension, all but two coefficients are held constant and the remaining two have a constant difference. Consider the polynomial $y = A_n x^n + A_{n-1}x^{n-1} + A_{n-2}x^{n-2} + \ldots + A_2x^2 + A_1x + A_0$. Choose 2 coefficients to vary, but keep their difference constant. All other coefficients will be held constant. Suppose $A_j - A_i = k, i < j$. If we have two polynomials in this family, we can determine where they will intersect by setting them equal to each other. Using a symbolic proof we see that the intersections will occur at the following points:

$$x^i = 0, y = A_0$$
$$x^{j-i} = -1, y = A_n(-1)^n + A_{n-1}(-1)^{n-1} + \ldots + A_0$$

However, depending on the values of i and j, these points may or may not be defined. Specifically, $x^{j-i} = -1$ will be defined as a real number only where j – i is an odd number.